On anisotropic Gauss-Bonnet cosmologies in (n+1) dimensions, governed by an n-dimensional Finslerian 4-metric

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Abstract

The $(n + 1)$-dimensional Einstein-Gauss-Bonnet (EGB) model is considered. For diagonal cosmological metrics, the equations of motion are written as a set of Lagrange equations with the effective Lagrangian containing two "minisuperspace" metrics on $\mathbb{R}^n$: a 2-metric of pseudo-Euclidean signature and a Finslerian 4-metric proportional to the n-dimensional Berwald-Moor 4-metric. For the case of the "pure" Gauss-Bonnet model, two exact solutions are presented, those with power-law and exponential dependences of the scale factors (w.r.t. the synchronous time variable). (The power-law solution was considered earlier by N. Deruelle, A. Toporensky, P. Tretyakov, and S. Pavluchenko.) In the case of EGB cosmology, it is shown that for any non-trivial solution with an exponential dependence of scale factors, $a_i(\tau) = A_i \exp(v^i \tau)$, there are no more than three different numbers among $v^1, ..., v^n$.

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1 Introduction

In this paper, we consider D-dimensional gravitational model with the Gauss-Bonnet term. The action reads

$$S = \int_M d^D z \sqrt{|g|} \{\alpha_1 R[g] + \alpha_2 \mathcal{L}_2[g]\}, \tag{1.1}$$

where $g = g_{MN} dz^M \otimes dz^N$ is a metric defined on the manifold M, $\dim M = D$, $|g| = |\det(g_{MN})|$, and

$$\mathcal{L}_2 = R_{MNPQ} R^{MNPQ} - 4 R_{MN} R^{MN} + R^2 \tag{1.2}$$

is the standard Gauss-Bonnet term. Here α_1 and α_2 are constants. The appearance of the Gauss-Bonnet term in multidimensional gravity is motivated by string theory [1, 2].

At present, the so-called Einstein-Gauss-Bonnet (EGB) gravity and its modifications are intensively used in cosmology, see [3, 4] (for $D = 4$), [6]-[12] and references therein, e.g., for explanation of the accelerated expansion of the Universe following from the supernovae (type Ia) observational data [13]. Certain exact solutions in multidimesional EGB cosmology were obtained in [6]-[12] and some other papers.

EGB gravity is also intensively investigated in the context of black-hole physics. The most important results here are related to the well-known Boulware-Deser-Wheeler solution [14] and its generalizations [15], for a review and references see [16]. (For certain applications of brane-world models with the Gauss-Bonnet term see also the review [17] and references therein.)

Here we are interested in cosmological solutions with diagonal metrics, governed by time-dependent scale factors.

For $\alpha_2 = 0$ we have the Kasner-type solution with the metric

$$g = -d\tau \otimes d\tau + \sum_{i=1}^{n} A_i^2 \tau^{2p^i} dy^i \otimes dy^i, \tag{1.3}$$

where $A_i > 0$ are arbitrary constants, $D = n + 1$, and parameters p^i obey the relations $\sum_{i=1}^{n} p^i = \sum_{i=1}^{n} (p^i)^2 = 1$ and hence $\sum_{1 \leq i < j \leq n} p^i p^j = 0$. For $D = 4$ it is the well-known Kasner solution [18].

In [5], the Einstein-Gauss-Bonnet (EGB) cosmological model was considered (see also [19]). For the "pure" Gauss-Bonnet (GB) case $\alpha_1 = 0$ and $\alpha_2 \neq 0$, N. Deruelle has obtained a cosmological solution with the metric (1.3) for $n = 4, 5$ and parameters obeying the relations

$$\sum_{i=1}^{n} p^i = 3, \qquad \sum_{1 \leq i < j < k < l \leq n} p^i p^j p^k p^l = 0. \tag{1.4}$$

It was reported by A. Toporensky and P. Tretyakov in [9] that this solution was verified by them for $n = 6, 7$. In the recent paper by S. Pavluchenko [20], the power-law solution was verified for all n (and also generalized to the Lowelock case [21]).

In this paper we give a derivation of the "power-law" solution (1.3), (1.4) and a solution with the exponential dependence of scale factors for arbitrary n. We note that the recent numerical analysis of cosmological solutions in EGB gravity for $D = 5, 6$ [11] shows that the singular solution (1.3), (1.4) (e.g. with a little generalization of the scale factors

$a_i(\tau) = A_i(\tau_0 \pm \tau)^{p^i}$, where τ_0 is constant) may appear as an asymptotical solution for certain initial values as well as the Kasner-type solution does.

The paper is organized as follows. In Section 2, the equations of motion for $(n+1)$-dimensional EGB model are considered. For diagonal cosmological metrics, the equations of motion are written in the form of Lagrange equations corresponding to a certain "effective" Lagrangian (see also [5, 20]). Section 3 deals with the "pure" Gauss-Bonnet model. Here two exact solutions are obtained, with power-law and exponential dependences of the scale factors on the synchronous time variable. In Section 4, the equations of motion are reduced to an autonomous set of first order differential equations in terms of synchronous time variable τ. For $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, we show that for any non-trivial solution with the exponential dependence of the scale factors $a_i(\tau) = A_i \exp(v^i \tau)$, $i = 1, ..., n$, there are no more than three different numbers among $v^1, ..., v^n$.

2 The cosmological model and its effective Lagrangian

2.1 The set-up

We consider the manifold
$$M = \mathbb{R}_* \times M_1 \times \ldots \times M_n, \tag{2.1}$$

with the metric
$$g = -e^{2\gamma(t)}dt \otimes dt + \sum_{i=1}^{n} e^{2\beta^i(t)}dy^i \otimes dy^i, \tag{2.2}$$

where M_i is a 1-dimensional manifold with the metric $g^i = dy^i \otimes dy^i$, $i = 1, \ldots, n$. Here and henceforth, $\mathbb{R}_* = (t_-, t_+)$ is an open subset in $\mathbb{R}$. (The functions $\gamma(t)$ and $\beta^i(t)$, $i = 1, \ldots, n$, are smooth on $\mathbb{R}_*$.)

The integrand in (1.1), if the metric (2.2) is substituted, reads (see [5] for $\gamma = 0$)

$$\sqrt{|g|}\{\alpha_1 R[g] + \alpha_2 \mathcal{L}_2[g]\} = L + \frac{df}{dt}, \tag{2.3}$$

where

$$L = \alpha_1 e^{-\gamma+\gamma_0} G_{ij}\dot{\beta}^i\dot{\beta}^j - \frac{1}{3}\alpha_2 e^{-3\gamma+\gamma_0} G_{ijkl}\dot{\beta}^i\dot{\beta}^j\dot{\beta}^k\dot{\beta}^l, \tag{2.4}$$

$\gamma_0 = \sum_{i=1}^{n} \beta^i$ and

$$G_{ij} = \delta_{ij} - 1, \tag{2.5}$$
$$G_{ijkl} = (\delta_{ij} - 1)(\delta_{ik} - 1)(\delta_{il} - 1)(\delta_{jk} - 1)(\delta_{jl} - 1)(\delta_{kl} - 1) \tag{2.6}$$

are the components of two "minisuperspace" metrics on $\mathbb{R}^n$. The first one is the well-known 2-metric of pseudo-Euclidean signature: $< v_1, v_2 > = G_{ij}v_1^i v_2^j$, while second one is the Finslerian 4-metric: $< v_1, v_2, v_3, v_4 > = G_{ijkl}v_1^i v_2^j v_3^k v_4^l$, $v_s = (v_s^i) \in \mathbb{R}^n$, where $< ., . >$ and $< ., ., ., . >$ are respectively 2- and 4-linear symmetric forms on $\mathbb{R}^n$. (We denote $\dot{A} = dA/dt$.)

The function $f = f(\gamma, \beta, \dot{\beta})$ in (2.3) is presented in Appendix B.

The derivation of (2.4) is based on the relations from Appendix A and the following identities

$$G_{ij}v^iv^j = \sum_{i=1}^{n}(v^i)^2 - (\sum_{i=1}^{n}v^i)^2, \tag{2.7}$$

$$G_{ijkl}v^iv^jv^kv^l = (\sum_{i=1}^{n}v^i)^4 - 6(\sum_{i=1}^{n}v^i)^2\sum_{j=1}^{n}(v^j)^2$$

$$+3(\sum_{i=1}^{n}(v^i)^2)^2 + 8(\sum_{i=1}^{n}v^i)\sum_{j=1}^{n}(v^j)^3 - 6\sum_{i=1}^{n}(v^i)^4. \tag{2.8}$$

It immediately follows from the definitions (2.5) and (2.6) that

$$G_{ij}v^iv^j = -2\sum_{i<j}v^iv^j, \tag{2.9}$$

$$G_{ijkl}v^iv^jv^kv^l = 24\sum_{i<j<k<l}v^iv^jv^kv^l. \tag{2.10}$$

Due to (2.10) $G_{ijkl}v^iv^jv^kv^l$ is zero for $n = 1, 2, 3$ ($D = 2, 3, 4$). For $n = 4$ ($D = 5$), $G_{ijkl}v^iv^jv^kv^l = 24v^1v^2v^3v^4$ and our 4-metric is proportional to the well-known Berwald-Moor 4-metric [23, 24] (see also [26, 27, 28] and references therein). We remind the reader that the 4-dimensional Berwald-Moor 4-metric obeys the relation: $< v, v, v, v >_{BM} = v^1v^2v^3v^4$. The Finslerian 4-metric with the components (2.6) coincides up to a factor with the n-dimensional analogue of the Berwald-Moor 4-metric. (This metric is a special case of 4th order Shimada metric [25].)

2.2   The equations of motion

The equations of motion corresponding to the action (1.1) have the form

$$\mathcal{E}_{MN} = \alpha_1\mathcal{E}^{(1)}_{MN} + \alpha_2\mathcal{E}^{(2)}_{MN} = 0, \tag{2.11}$$

where

$$\mathcal{E}^{(1)}_{MN} = R_{MN} - \frac{1}{2}Rg_{MN}, \tag{2.12}$$

$$\mathcal{E}^{(2)}_{MN} = 2(R_{MPQS}R_N{}^{PQS} - 2R_{MP}R_N{}^P$$

$$-2R_{MPNQ}R^{PQ} + RR_{MN}) - \frac{1}{2}\mathcal{L}_2g_{MN}. \tag{2.13}$$

The field equations (2.11) for the metric (2.2) are equivalent to the Lagrange equations corresponding to the Lagrangian L from (2.4) [5].

Thus eqs. (2.11) read

$$\alpha_1G_{ij}\dot{\beta}^i\dot{\beta}^j - \alpha_2e^{-2\gamma}G_{ijkl}\dot{\beta}^i\dot{\beta}^j\dot{\beta}^k\dot{\beta}^l = 0, \tag{2.14}$$

$$\frac{d}{dt}[2\alpha_1G_{ij}e^{-\gamma+\gamma_0}\dot{\beta}^j - \frac{4}{3}\alpha_2e^{-3\gamma+\gamma_0}G_{ijkl}\dot{\beta}^j\dot{\beta}^k\dot{\beta}^l] - L = 0, \tag{2.15}$$

$i = 1, \ldots, n$. Due to (2.14),

$$L = \frac{2}{3}e^{-\gamma+\gamma_0}\alpha_1G_{ij}\dot{\beta}^i\dot{\beta}^j. \tag{2.16}$$

3 Exact solutions in "pure" Gauss-Bonnet model

Now we put $\alpha_1 = 0$ and $\alpha_2 \neq 0$, i.e. we consider the cosmological model governed by the action

$$S_2 = \alpha_2 \int_M d^D z \sqrt{|g|} \mathcal{L}_2[g]. \tag{3.1}$$

The equations of motion (2.11) in this case read

$$\mathcal{E}_{MN}^{(2)} = \mathcal{R}_{MN}^{(2)} - \frac{1}{2}\mathcal{L}_2 g_{MN} = 0, \tag{3.2}$$

where

$$\mathcal{R}_{MN}^{(2)} = 2(R_{MPQS}R_N{}^{PQS} - 2R_{MP}R_N{}^P \\ -2R_{MPNQ}R^{PQ} + RR_{MN}). \tag{3.3}$$

Due to the identity $g^{MN}\mathcal{R}_{MN}^{(2)} = 2\mathcal{L}_2$ the set of eqs. (3.2) for $D \neq 4$ implies

$$\mathcal{L}_2 = 0. \tag{3.4}$$

It is obvious that the set of equations (3.2) is equivalent for $D \neq 4$ to the set of equations

$$\mathcal{R}_{MN}^{(2)} = 0. \tag{3.5}$$

The equations of motion (2.14), (2.15) in this case read

$$G_{ijkl}\dot{\beta}^i \dot{\beta}^j \dot{\beta}^k \dot{\beta}^l = 0, \tag{3.6}$$

$$\frac{d}{dt}\left[e^{-3\gamma+\gamma_0} G_{ijkl}\dot{\beta}^j \dot{\beta}^k \dot{\beta}^l \right] = 0, \tag{3.7}$$

$i = 1, \ldots, n$. Here $L = 0$ due to (3.6).

Let us put $\ddot{\beta}^i = 0$ for all i, or equivalently,

$$\beta^i = c^i t + c_0^i, \tag{3.8}$$

where c^i and c_0^i are constants, $i = 1, \ldots, n$.

We also put

$$3\gamma = \gamma_0 = \sum_{i=1}^n \beta^i, \tag{3.9}$$

i.e. a modified "harmonic" time variable is used. Recall that in the case $\alpha_1 \neq 0$ and $\alpha_2 = 0$, the choice $\gamma = \gamma_0$ corresponds to the harmonic time variable t [31].

Then, eqs. (3.7) are satisfied identically and eq. (3.6) gives us the following constraint

$$G_{ijkl}c^i c^j c^k c^l = 24 \sum_{i<j<k<l} c^i c^j c^k c^l = 0. \tag{3.10}$$

Thus we have obtained an exact cosmological solution for the Gauss-Bonnet model (3.1), given by the metric (2.2) with the functions $\beta^i(t)$ and $\gamma(t)$ from (3.8) and (3.9), respectively, and the integration constants c^i obeying (3.10).

3.1 Power-law solutions

Let us consider the solutions with $\sum_{i=1}^{n} c^i \neq 0$.

Introducing the synchronous time variable $\tau = \frac{1}{c} \exp(ct + c_0)$, where $c = \frac{1}{3} \sum_{i=1}^{n} c^i$, $c_0 = \frac{1}{3} \sum_{i=1}^{n} c_0^i$, and defining the new parameters $p^i = c^i/c$, $A_i = \exp[c_0^i + p^i(\ln c - c_0)]$, we get the power-law solution with the metric

$$g = -d\tau \otimes d\tau + \sum_{i=1}^{n} A_i^2 \tau^{2p^i} dy^i \otimes dy^i, \tag{3.11}$$

where $A_i > 0$ are arbitrary constants, and the parameters p^i obey the relations

$$\sum_{i=1}^{n} p^i = 3, \tag{3.12}$$

$$G_{ijkl} p^i p^j p^k p^l = 24 \sum_{i<j<k<l} p^i p^j p^k p^l = 0. \tag{3.13}$$

This solution is singular for any set of parameters [22]. For $n = 4, 5$ it was obtained in [5].

Example 1. Let $D = 6$ and $p_i \neq 0$, $i = 1, \ldots, 5$. Then the relations (3.12) and (3.13) read

$$p^1 + p^2 + p^3 + p^4 + p^5 = 3, \tag{3.14}$$

$$p^1 p^2 p^3 p^4 p^5 \left(\frac{1}{p^1} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \frac{1}{p^5} \right) = 0. \tag{3.15}$$

Let us put $p^1 = x > 0$, $p^2 = \frac{1}{x}$, $p^3 = z > 0$, $p^4 = y < 0$, $p^5 = \frac{1}{y}$. Then we get

$$x + \frac{1}{x} + z + y + \frac{1}{y} = 3, \quad x + \frac{1}{x} + \frac{1}{z} + y + \frac{1}{y} = 0. \tag{3.16}$$

Subtracting the second relation in (3.16) from the first one, we obtain $z - \frac{1}{z} = 3$ or $z = \frac{1}{2}(3 + \sqrt{13})$ $(z > 0)$. For any $x > 0$ there are two solutions $y = y_\pm(x) = \frac{1}{2}(-A \pm \sqrt{A^2 - 4})$, where $A = x + \frac{1}{x} + \frac{1}{z} > 2$.

Proposition. *For $D > 4$, the metric (3.11) is a solution to eqs. of motion (3.2) if and only if the set of parameters $p = (p^1, ..., p^n)$ either obeys the relations (3.12) and (3.13), or $p = (a, b, 0, ..., 0), (a, 0, b, 0, ..., 0), \ldots$, where a, b are arbitrary real numbers.*

This proposition is proved in Appendix C (for $D = 5, 6$ see also [5]).

For $D = 2, 3, 4$ the metric (3.11) gives a solution to the equations of motion (3.2) for any set of parameters p^i.

3.2 Exponential solutions

Now we consider solutions with $\sum_{i=1}^{n} c^i = 0$. Introducing the synchronous time variable $\tau = t \exp(c_0)$ where $c_0 = \frac{1}{3} \sum_{i=1}^{n} c_0^i$ and defining the new parameters $v^i = c^i \exp(-c_0)$, $B_i = \exp(c_0^i)$, we get a non-singular cosmological solution with the metric

$$g = -d\tau \otimes d\tau + \sum_{i=1}^{n} B_i^2 e^{2v^i \tau} dy^i \otimes dy^i, \tag{3.17}$$

where $B_i > 0$ are arbitrary constants, and parameters v^i obey the relations

$$\sum_{i=1}^{n} v^i = 0, \tag{3.18}$$

$$G_{ijkl} v^i v^j v^k v^l = 24 \sum_{i<j<k<l} v^i v^j v^k v^l = 0. \tag{3.19}$$

Example 2. Let $D = 6$ and $v_i \neq 0$, $i = 1, \ldots, 5$. The relations (3.18) and (3.19) read in this case

$$v^1 + v^2 + v^3 + v^4 + v^5 = 0, \tag{3.20}$$

$$v^1 v^2 v^3 v^4 v^5 \left(\frac{1}{v^1} + \frac{1}{v^2} + \frac{1}{v^3} + \frac{1}{v^4} + \frac{1}{v^5} \right) = 0. \tag{3.21}$$

We put $v^1 = x > 0$, $v^2 = \frac{1}{x}$, $v^3 = 1$, $v^4 = y < 0$, $v^5 = \frac{1}{y}$. Then we get

$$x + \frac{1}{x} + 1 + y + \frac{1}{y} = 0, \tag{3.22}$$

For any $x > 0$ there are two solutions of (3.22): $y = y_{\pm}(x) = \frac{1}{2}(-B \pm \sqrt{B^2 - 4})$, where $B = x + \frac{1}{x} + 1 \geq 3$.

Remark. For $D = 4$, or $n = 3$, the equations of motion (3.6), (3.7) are satisfied identically for arbitrary (smooth) functions $\beta^i(t)$ and $\gamma(t)$. This is in an agreement with the fact that in dimension $D = 4$ the action (3.1) is a topological invariant, and its variation vanishes identically.

4 Reduction to an autonomous set of first-order differential equations

Now we put $\gamma = 0$, i.e. the "synchronous" time gauge is considered. We denote $t = \tau$. By introducing the "Hubble-like" variables $h^i = \dot{\beta}^i$ we rewrite eqs. (2.14) and (2.15) in the following form:

$$\alpha_1 G_{ij} h^i h^j - \alpha_2 G_{ijkl} h^i h^j h^k h^l = 0, \tag{4.1}$$

$$\left[2\alpha_1 G_{ij} h^j - \frac{4}{3} \alpha_2 G_{ijkl} h^j h^k h^l \right] \sum_{s=1}^{n} h^s$$

$$+ \frac{d}{d\tau} \left[2\alpha_1 G_{ij} h^j - \frac{4}{3} \alpha_2 G_{ijkl} h^j h^k h^l \right] - L = 0, \tag{4.2}$$

$i = 1, \ldots, n$, where

$$L = \alpha_1 G_{ij} h^i h^j - \frac{1}{3} \alpha_2 G_{ijkl} h^i h^j h^k h^l, \tag{4.3}$$

see also [5].

Due to (4.1), $L = \frac{2}{3} \alpha_1 G_{ij} h^i h^j$. Thus we obtain an autonomous set of first order differential equations with respect to $h^1(\tau), \ldots, h^n(\tau)$.

Here we may use the relations (2.7), (2.8) and the following formulae (with $v^i = h^i$)

$$G_{ij}v^j = v^i - S_1, \qquad (4.4)$$

$$G_{ijkl}v^j v^k v^l = S_1^3 + 2S_3 - 3S_1 S_2 + 3(S_2 - S_1^2)v^i + 6S_1(v^i)^2 - 6(v^i)^3, \qquad (4.5)$$

$i = 1, \ldots, n$, where $S_k = S_k(v) = \sum_{i=1}^n (v^i)^k$.

Let us consider a fixed point of the system (4.1), (4.2): $h^i(\tau) = v^i$, where the constant vector $v = (v^i)$ corresponds to the solution

$$\beta^i = v^i \tau + \beta_0^i, \qquad (4.6)$$

β_0^i are constants, and $i = 1, \ldots, n$. In this case we obtain the metric (3.17) with the exponential dependence of the scale factors. (Another solution with $h^i(\tau) = p^i/\tau$ was obtained earlier in subsection 3.2.)

Now we put $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. For fixed the point $v = (v^i)$ we have the set of polynomial equations

$$G_{ij}v^i v^j - \alpha G_{ijkl}v^i v^j v^k v^l = 0, \qquad (4.7)$$

$$\left[2G_{ij}v^j - \frac{4}{3}\alpha G_{ijkl}v^j v^k v^l\right] \sum_{s=1}^n v^s - \frac{2}{3}G_{ij}v^i v^j = 0, \qquad (4.8)$$

$i = 1, \ldots, n$, where $\alpha = \alpha_2/\alpha_1$. For $n > 3$ it is a set of forth-order polynomial equations.

The trivial solution $v = (v^i) = (0, \ldots, 0)$ corresponds to a flat metric g.

For any nontrivial solution v we have $\sum_{i=1}^n v^i \neq 0$ (otherwise one gets from (4.8) $G_{ij}v^i v^j = \sum_{i=1}^n (v^i)^2 - (\sum_{i=1}^n v^i)^2 = 0$ and hence $v = (0, \ldots, 0)$).

Let us consider the isotropic case $v^1 = \ldots = v^n = a$. The set of equations (4.7), (4.8) is equivalent to the equation

$$n(n-1)a^2 + \alpha n(n-1)(n-2)(n-3)a^4 = 0. \qquad (4.9)$$

For $n = 1$, a is arbitrary, and $a = 0$ for $n = 2, 3$. If $n > 3$ the nonzero solution to eq. (4.9) exists only if $\alpha < 0$, and in this case

$$a = \pm \frac{1}{\sqrt{|\alpha|(n-2)(n-3)}}. \qquad (4.10)$$

Here arises the problem of classification of all solutions to eqs. (4.7), (4.8) for given n. Some special solutions of the form $(a, \ldots, a, b, \ldots, b)$, e.g., in the context of cosmology with two factor spaces, for certain dimensions were considered in literature, see, e.g., [6, 7, 8, 12].

Let us outline three properties of the solutions to the set of polynomial equations (4.7), (4.8):

i) For any solution $v = (v^1, \ldots, v^n)$, the vector $(-v) = (-v^1, \ldots, -v^n)$ is also a solution;

ii) For any solution $v = (v^1, \ldots, v^n)$ and for any permutation σ of the set of indices $\{1, \ldots, n\}$, the vector $v = (v^{\sigma(1)}, \ldots, v^{\sigma(n)})$ is also a solution;

iii) For any nontrivial solution $v = (v^1, \ldots, v^n) \neq (0, \ldots, 0)$ there are no more than three different numbers among $v^1, \ldots, v^n$.

The first proposition is trivial. The second one simply follows from the relations (2.7), (2.8), (4.4), (4.5).

Let us prove the third proposition. Suppose that there exists a nontrivial solution $v = (v^1, ..., v^n)$ with more than three different numbers among $v^1, ..., v^n$. Due to (4.5), (4.8) and $\sum_{i=1}^{n} v^i \neq 0$, any number v^i obeys the cubic equation $C_0 + C_1 v^i + C_2 (v^i)^2 + C_3 (v^i)^3 = 0$, with $C_3 \neq 0$, $i = 1, \ldots, n$, and hence at most three numbers among v^i may be different. Thus we obtain a contradiction. The proposition **iii)** is proved.

This implies that in future investigations of solutions to eqs. (4.7), (4.8) for arbitrary n we will to consider three nontrivial cases such that: 1) $v = (a, ..., a)$ (see (4.10)); 2) $v = (a, ..., a, b, ..., b)$ $(a \neq b)$; and 3) $v = (a, ..., a, b, ..., b, c, ..., c)$ $(a \neq b, b \neq c, a \neq c)$. One may also put $a > 0$ due to item **i)**.

5   Conclusions and discussions

We have considered the $(n + 1)$-dimensional Einstein-Gauss-Bonnet model. For diagonal cosmological metrics, we have written the equations of motion as a set of Lagrange equations (see also [5]) with the effective Lagrangian governed by two "minisuperspace" metrics on $\mathbb{R}^n$: (i) the pseudo-Euclidean 2-metric (corresponding to the scalar curvature term) and (ii) the Finslerian 4-metric (corresponding to the Gauss-Bonnet term). The Finslerian 4-metric is proportional to the n-dimensional Berwald-Moor 4-metric (it is a special case of the Shimada quartic metric [25]). Thus we have found rather a natural and "legitimate" application of the n-dimensional Berwald-Moor metric ($n = 4, 5, \ldots$) in $(n + 1)$-dimensional gravity with a Gauss-Bonnet term. The effective Lagrangian (2.4) was considered earlier by N. Deruelle in [5] (for $\gamma = 0$). (See also [20] and references therein.) Here we put an additional accent on the Finslerian (Berwald-Moor) structure of the second term in (2.4).

For the case of the pure Gauss-Bonnet model, we have derived two exact solutions with power-law and exponential dependences of the scale factors on the synchronous time variable. The first (power-law) solution was obtained earlier by N. Deruelle for $n = 4, 5$ [5] and verified by A. Toporensky and P. Tretyakov for $n = 6, 7$ [9]. In [20] this solution was verified for all n.

When the synchronous time gauge was considered, the equations of motion were reduced to an autonomous set of first-order differential equations (see also [5]). For $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ it was shown that for any nontrivial solution with the exponential time dependence of the scale factors $a_i(\tau) = A_i \exp(v^i \tau)$, there are no more than three different numbers among $v^1, ..., v^n$. This means that solutions of this type have a restricted anisotropy. Such solutions may be used for construction of new cosmological solutions, e.g., describing an accelerated expansion of our 3-dimensional factor space and small enough variation of the effective gravitational constant. For this approach see [29, 30] and references therein.

Here an open problem arises: do the solutions (1.3), (1.4) with "jumping" parameters p^i, A_i appear as asymptotic solutions in EGB gravity (for some n) when approaching a singular point? Recall that the Kasner-type solutions with "jumping" parameters p^i, A_i describe an approach to a singular point in certain gravitational models, e.g., with matter sources, see [32, 33, 34, 35, 36, 37, 38] and references therein. This problem may be a subject of separate investigations. [Worth mentioning is the paper by T. Damour and H. Nicolai [39], which includes a study of the effect of the 4th order (in curvature) gravity terms, including the Euler-Lovelock term (octic in velocities), and its compatibility with

the Kac-Moody algebra E_{10}.]

Other applications of the Lagrange approach considered in [5] and in the present paper will be connected with inclusion of scalar fields and a generalization to the Lowelock model [21] (see also [5, 20]).

Appendix

A $(1+n)$-splitting

Consider the metric defined on $\mathbb{R}_* \times \mathbb{R}^n$ ($\mathbb{R}_* = (t_-, t_+)$ is an open subset in $\mathbb{R}$)

$$g = -e^{2\gamma(t)}dt \otimes dt + \sum_{i,j=1}^{n} h_{ij}(t)dy^i \otimes dy^j. \tag{A.1}$$

Here $(h_{ij}(t))$ is a symmetric nondegenerate matrix for any $t \in \mathbb{R}_*$, smoothly depending on u. The function $\gamma(t)$ is smooth.

Calculations give the following nonvanishing (identically) components of the Riemann tensor

$$R_{0i0j} = -R_{i00j} = -R_{0ij0} = R_{i0j0} = \frac{1}{4}[-2\ddot{h}_{ij} + 2\dot{\gamma}\dot{h}_{ij} + \dot{h}_{ik}h^{kl}\dot{h}_{lj}], \tag{A.2}$$

$$R_{ijkl} = \frac{1}{4}e^{-2\gamma}(\dot{h}_{ik}\dot{h}_{jl} - \dot{h}_{il}\dot{h}_{jk}), \tag{A.3}$$

$i, j, k, l = 1, \ldots, n$, where $h^{-1} = (h^{ij})$ is the matrix inverse to the matrix $h = (h_{ij})$. We denote $\dot{A} = dA/dt$ etc.

For nonzero (identically) components of the Ricci tensor we get

$$R_{00} = \frac{1}{2}[-h^{il}\ddot{h}_{li} + \frac{1}{2}h^{ij}\dot{h}_{jk}h^{kl}\dot{h}_{li} + h^{ik}\dot{h}_{ki}\dot{\gamma}], \tag{A.4}$$

$$R_{ij} = \frac{1}{4}e^{-2\gamma}[2\ddot{h}_{ij} + \dot{h}_{ij}(h^{kl}\dot{h}_{lk} - 2\dot{\gamma}) - 2\dot{h}_{ik}h^{kl}\dot{h}_{lj}], \tag{A.5}$$

$i, j = 1, \ldots, n$.

The scalar curvature reads

$$R = \frac{1}{4}e^{-2\gamma}[4\mathrm{tr}(\ddot{h}h^{-1}) + \mathrm{tr}(\dot{h}h^{-1})(\mathrm{tr}(\dot{h}h^{-1}) - 4\dot{\gamma}) - 3\mathrm{tr}(\dot{h}h^{-1}\dot{h}h^{-1})]. \tag{A.6}$$

B f-function

The function f in (2.3) has the following form (see [5] for $\gamma = 0$)

$$f = \alpha_1 f_1 + \alpha_2 f_2, \tag{B.1}$$

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where

$$f_1 = 2e^{-\gamma+\gamma_0} \sum_{i=1}^{n} \dot{\beta}^i, \qquad (B.2)$$

$$f_2 = \frac{4}{3}e^{-3\gamma+\gamma_0} \left[2\sum_{i=1}^{n}(\dot{\beta}^i)^3 - 3(\sum_{i=1}^{n}\dot{\beta}^i)\sum_{j=1}^{n}(\dot{\beta}^i)^2 + (\sum_{i=1}^{n}\dot{\beta}^i)^3 \right]. \qquad (B.3)$$

The function f_2 may be rewritten as follows:

$$f_2 = \frac{4}{3}e^{-3\gamma+\gamma_0}G_{ijk}\dot{\beta}^i\dot{\beta}^j\dot{\beta}^k, \qquad (B.4)$$

where

$$G_{ijk} = (\delta_{ij} - 1)(\delta_{ik} - 1)(\delta_{jk} - 1) \qquad (B.5)$$

are components of a Finslerian 3-metric.

C   Proof of the Proposition in Sec. 3.1

The equations of motion (4.1) and (4.2) corresponding to the metric (3.11) with $h^i = p^i/\tau$ (here $\alpha_1 = 0$ and $\alpha_2 \neq 0$) read

$$\mathcal{A} \equiv G_{ijkl}p^i p^j p^k p^l = 0, \qquad (C.1)$$

$$\mathcal{D}_i \equiv G_{ijkl}p^j p^k p^l = 0, \qquad (C.2)$$

$i = 1, \ldots, n$.

Let $D = n + 1 \neq 4$ and

$$\mathcal{B} \equiv \frac{1}{(n-3)} \sum_{i=1}^{n} \mathcal{D}_i, \qquad \mathcal{C}_i \equiv \frac{1}{3}(\mathcal{B} - \mathcal{D}_i), \qquad (C.3)$$

$i = 1, \ldots, n$.

For $D \neq 4$ the set of eqs. (C.1), (C.2) is equivalent to the following set of equations:

$$\mathcal{A} = S_1^4 - 6S_1^2 S_2 + 3S_2^2 + 8S_1 S_3 - 6S_4 = 24 \sum_{i<j<k<l} p^i p^j p^k p^l = 0, \qquad (C.4)$$

$$\mathcal{B} = (S_1 - 3)(S_1^3 - 3S_1 S_2 + 2S_3) = 6(S_1 - 3) \sum_{i<j<k} p^i p^j p^k = 0, \qquad (C.5)$$

$$\mathcal{C}_i = (S_1 - 3)p^i[2(p^i)^2 - 2S_1 p^i + S_1^2 - S_2] = 0, \qquad (C.6)$$

$i = 1, \ldots, n$. Here $S_k = S_k(p) = \sum_{i=1}^{n}(p^i)^k$, and we have used the identities (2.8), (4.5) and the following identity:

$$S_1^3 - 3S_1 S_2 + 2S_3 = G_{ijk}p^i p^j p^k = 6 \sum_{i<j<k} p^i p^j p^k, \qquad (C.7)$$

where G_{ijk} are defined in (B.5).

For $S_1 = 3$ we obtain the main solution governed by the relations (3.12) and (3.13).

Now consider another case, $S_1 \neq 3$. Let k be the number of all nonzero numbers among $p^1, ..., p^n$. For $k = 0$ we get the trivial solution $(0, ..., 0)$. Let $k \geq 1$. We suppose without loss of generality that $p^1, ..., p^k$ are nonzero. For $k = 1, 2$ all relations (C.4)-(C.6) are satisfied identically. In all three cases $k = 0, 1, 2$ the solutions have the form $(a, b, 0.., 0)$ (plus permutations for the general setup).

Consider $k \geq 3$. From (C.6) and $S_1 \neq 3$ we obtain

$$2(p^i)^2 - 2S_1 p^i + S_1^2 - S_2 = 0, \tag{C.8}$$

$i = 1, \ldots, k$. Summing over i gives us $(2 - k)(S_2 - S_1^2) = 0$, or $S_2 = S_1^2$. Then we obtain from (C.5) $S_3 = S_1^3$ and from (C.4): $S_4 = S_1^4$. Thus we get $S_4 = S_2^2$ implying $\Sigma = \sum_{1 \leq i < j \leq k} (p^i)^2 (p^j)^2 = 0$. But $\Sigma \geq (p^1)^2 (p^2)^2 > 0$. Thus we obtain a contradiction. That means that for $S_1 \neq 3$ we have only solutions with $k \leq 2$ of the form $(a, b, 0.., 0)$ (plus permutations for general setup). The Proposition in Subsection 3.1 is proved.

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